Goal: <u>General Construction of all irreducible representations of a s.s. Lie algebra:</u>

<u>Redo for sl_2</u>

<u>sl_2</u>: $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $[E,F] = H$, $[H,E] = 2E$, $[H,F] = -2F$

$\mathfrak{h} = \mathbb{C}H$

Let V be an sl_2 module:

ⓐ a weight vector = eigenvector of H.

ⓑ v higher weight vector if $Ev = 0$

<u>Lemma</u> Let $v \in V$ be a weight vector with weight μ (i.e. $Hv = \mu v$)

$\Rightarrow$ $H(Ev) = (\mu+2)(Ev)$

$H(Fv) = (\mu - 2)(Fv)$

<u>Lemma</u>: Assume $H \in \mathrm{End}(V)$, H diagonalizable (not necessarily element in sl_2)

Let $W \subset V$ be an H-submodule (i.e. $w \in W \Rightarrow Hw \in W$)

ⓐ Assume $w \in W$, $w = \sum_{i=1}^{r} \alpha_i v_i$ with $Hv_i = \lambda_i v_i$, and $\lambda_i \neq \lambda_j$ for $i \neq j$

$\Rightarrow v_i \in W \quad \forall i$ with $\alpha_i \neq 0$

ⓑ If v_i eigenvectors of H with eigenvalues λ_i, $\lambda_i \neq \lambda_j$ for $i \neq j$ $\Rightarrow$ $\{v_i\}$ lin. independent)

Proof by ind. on r

$r = 1$. $w = v_1$ eigenvector ✓

$r \to r+1$ ~~Assume~~ Let $w_1 = (H - \lambda_{r+1})w = \sum_{i=1}^{r+1} \alpha_i \underbrace{(\lambda_i - \lambda_{r+1})}_{=0 \text{ for } i = r+1} v_i$

$= \sum_{i=1}^{r} \alpha_i (\lambda_i - \lambda_{r+1}) v_i$

$\Rightarrow v_1, \dots v_r \in W$

by induction assumption

$\Rightarrow v_{r+1} = \frac{1}{\alpha_{r+1}} \big(\underset{\in W}{w} - \underbrace{\sum_{i=1}^{r} \alpha_i v_i}_{\in W} \big) \in W$

Theorem: Let $\lambda \in \mathbb{C}$, ~~V a vector~~

$\Rightarrow \exists$ $\mathfrak{sl}_2$-module M_λ with countable basis $\{v_0, v_1, v_2 \ldots\}$ s.t. action of $\mathfrak{sl}_2$ is given by

$$Hv_i = (\lambda - 2i) v_i \qquad \leftarrow \text{all weights } \leq \lambda$$

$$Fv_i = (i+1) v_{i+1}$$

$$Ev_i = (\lambda + 1 - i) v_{i-1}, \quad \text{with } v_{-1} = 0. \quad (\text{i.e. } Ev_0 = 0)$$

(b) M_λ has a submodule $\neq 0 \iff \lambda \in \mathbb{Z}_{\geq 0}$

(a) Any $\mathfrak{sl}_2$-module ~~gen~~ M generated by a h.w. vector with weight λ is a quotient of M_λ

Proof.

To show that M_λ is an $\mathfrak{sl}_2$-module, it suffices to check the relations for each vector v_i. This is straightforward

e.g. $EFv_i = (i+1) Ev_{i+1} = (\lambda - i)(i+1) v_i$

$FEv_i = F(\lambda+1-i) v_{i-1} = i(\lambda+1-i) v_i$

$$\Rightarrow \underbrace{(EF - FE)}_{[E,F]} v_i = \big((\lambda - i)(i+1) - (\lambda + 1 - i) i\big) v_i = \big(\lambda(i+1-i) - i(i+1) + i(1-i)\big) v_i$$

$$= (\lambda - i^2 - i - i + i^2) v_i$$

$$= (\lambda - 2i) v_i$$

$$= Hv_i$$

Remark: More efficient way to prove this done later.

Now let M be any $\mathfrak{sl}_2$-module generated by h.w. vector $w_0 \in M$ ~~s.t.~~ with weight λ

Define: ~~let~~ inductively

$$w_{i+1} = \frac{1}{i+1} F w_i$$

claim: $M = \operatorname{span}\{w_i, w_i \neq 0\} = \tilde{M}$

proof: enough to show: $Xw_i \in \tilde{M} \ \forall w_i \in \tilde{M}$

$X = F$ ✓

$Hw_i = (\lambda - 2i) w_i$ easy induction.

$$E w_i = \begin{cases} (\lambda+1-i) w_{i-1} & i>0 \\ 0 & i=0 \end{cases}$$

again by induction, wile $i=0$ ✓

$$E w_{i+1} = EF\left(\tfrac{1}{i+1} w_i\right) = \frac{1}{i+1}(H+FE)w_i$$

$$= \frac{1}{i+1}(\lambda-2i + \cancel{(i+1)(\lambda} \; i(\lambda+1-i) w_i$$

$$= \frac{1}{i+1}\left((i+1)\lambda \underbrace{-2i+i-i^2}_{-i(i+1)} \right) w_i$$

$$= (\lambda - i) w_i$$

$$= (\lambda+1-(i+1)) w_i$$

As $w_0 \in \tilde{M}$ generates $M \Rightarrow \tilde{M} = M$

<u>claim</u> $\Rightarrow$ $\Phi : M_\lambda \to M$

$v_i \to w_i$ defines map of sl_2-modules

(there

(if $\dim M$ is finite, $\dim M = n+1$ we define $w_i = 0$ for $i > n$)

ⓐ If $\dim M = \infty \Rightarrow \Phi$ defines an isom. as actions are exactly the same

ⓑ assume $\dim M = n+1$ ~~Let $n = \max\{i, w_i \neq 0\}$~~ $\Rightarrow Fw_n = 0$, and $w_n \neq 0$

[reason $w_i = 0$
$\Rightarrow Fw_i = (i+1) w_{i+1} = 0$
$\Rightarrow w_j = 0 \;\forall j > i$]

$$\Rightarrow H w_n = (\lambda - 2n) w_n$$

$$\parallel$$

$$(EF-FE)w_n \underset{\substack{\uparrow \\ Fw_n = 0}}{=} -FEw_n = -F(\lambda+1-n)w_{n-1} = -n(\lambda+1-n)w_n$$

$$\Rightarrow \lambda - 2n = -n\lambda - n + n^2$$

$$(n+1)\lambda = n^2 + n = n(n+1)$$

$$\Rightarrow \boxed{\lambda = n}$$

$\Rightarrow$ v_{n+1} is a highest weight vector with weight $\lambda - 2(n+1) = -n-2$.

$\Rightarrow$ $\mathfrak{sl}_2$ module generated by v_{n+1}

$= \text{span}\{v_{n+1}, v_{n+2}, v_{n+3} \ldots\}$

$\cong M_{-n-2}$

and $M_n / M_{-n-2} \cong M = (n+1)\text{-dim irred. representation}$

Finally, assume $W \subset M_\lambda$ is a submodule
Let $w = \sum d_i v_i$ be in W
$\uparrow$ only finitely many $d_i \neq 0$

Lemma $\Rightarrow$ $v_i \in W$ for some i.

~~if $i=0 \Rightarrow v_0 \in W$~~

if $\lambda \notin \mathbb{Z}_{\geq 0}$ $\Rightarrow$ $\lambda + 1 - i \neq 0$ $\quad (\Leftarrow \lambda = i-1 \in \mathbb{Z}_{\geq 0}$ for $i \neq 0)$

~~if $i=0$~~ $\Rightarrow$ $v_{i-1} = \frac{1}{\lambda+1-i} E v_i \in W$

$\Rightarrow \cdots$ $v_0 \in W$

$\Rightarrow$ $W = M_\lambda$ as v_0 generates M_λ

if $i=0 \Rightarrow v_i = v_0 \Rightarrow W = M_\lambda$

hence M_λ has no submodule for $\lambda \neq n \geq 0$

same argument: if $\lambda = n \geq 0$

only submodule of M_λ is $\text{span}\{v_{n+1}, v_{n+2} \ldots\} \cong M_{-n-2}$
$\uparrow$ simple

Details: If we can find a v_i in W with i<n+1, then also v_0 in W and hence W=M_lambda.
If we can only find v_i in W with i>n, then v_{n+1} in W, which is a highest weight vector